Inference for the multivariate coefficients of variation in factorial designs

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- The coefficient of variation (CV)

$$c = \frac{\sigma}{\mu}$$

  is a widely used unit-free measure of dispersion.
- It is a popular tool to judge, e.g.,
  - the repeatability of measurements in clinical trials,
  - the risk in the financial world or in psychology,
  - the quantitative variability in genetics.
- It is also a reliability tool in control charts for monitoring.
- Various inference methods are suggested to compare two or several groups in terms of CV (Aerts and Haesbroeck, 2017; Pauly and S., 2020).

- However, when more than one feature is of interest, comparisons based on marginal CVs are misleading due to potentially different decisions for the single features (Van Valen, 1974), and does not account for correlations between the features.
- The solution is to use the multivariate coefficient of variation (MCV).
- However, the extension is not unique and there is no default choice up until now.
- Reyment (1960), Van Valen (1974), Voinov and Nikulin (1996) and Albert and Zhang (2010) suggest to define the MCV by

$$C^{\mathrm{RR}} = \sqrt{\frac{(\det \boldsymbol{\Sigma})^{1/d}}{\boldsymbol{\mu}^\top \boldsymbol{\mu}}}, \; C^{\mathrm{VV}} = \sqrt{\frac{\mathrm{tr}\boldsymbol{\Sigma}}{\boldsymbol{\mu}^\top \boldsymbol{\mu}}}, \; C^{\mathrm{VN}} = \sqrt{\frac{1}{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}}, \; C^{\mathrm{AZ}} = \sqrt{\frac{\boldsymbol{\mu}^\top \boldsymbol{\Sigma} \boldsymbol{\mu}}{(\boldsymbol{\mu}^\top \boldsymbol{\mu})^2}},$$

respectively. Here $\boldsymbol{\mu}$ denotes the nonzero mean vector of a d-dimensional random variable and $\boldsymbol{\Sigma}$ is corresponding covariance matrix.

- All these definitions reduces to the CV in the univariate $(d = 1)$ case.
- The differences of them are discussed in great detail by Albert and Zhang (2010).
- Additionally, the standardized means as the reciprocal of C^v are of their own interest:

$$B^v = \frac{1}{C^v} \quad (v = \text{RR, VV, VN, AZ}). \tag{1}$$

- A further problem of the MCV is the lack of generally applicable inference methods.
- Aerts and Haesbroeck (2017) considered testing the equality of several MCVs following the definition of Voinov and Nikulin (1996):

$$C^{\mathsf{VN}} = \sqrt{\frac{1}{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}}.$$

- But their methods rely on the specific assumption of the underlying distribution (elliptical symmetric distribution) and the convergence speed of their test statistic is rather slow leading to an inconsistent type-1 error control for small sample sizes.
- Ditzhaus and S. (2022) proposed the permutation test for the above MCV in factorial designs, which has good theoretical and finite sample properties.

- We consider n_1 independent, identically distributed d-dimensional random variables

$$\boldsymbol{X}_j = (X_{j1}, \ldots, X_{jd})^\top \quad (j = 1, \ldots, n_1).$$

- We suppose no specific conditions on the distributions of $\boldsymbol{X}_j$ except the following assumptions on the moments to ensure the well-definedness of C^j and B^j:

Assumption 1

Let $\boldsymbol{\mu} \neq \mathbf{0}$ and $E(X_{j\ell}^4) < \infty$ for all j and ℓ. Moreover, we suppose:

a. *For C^{RR}, C^{VN}, B^{RR}, and B^{VN}, we consider only regular matrices $\boldsymbol{\Sigma}$.*
b. *For C^{VV} and B^{VV}, we assume $\boldsymbol{\Sigma} \neq \mathbf{0}_{d \times d}$.*
c. *For C^{AZ} and B^{AZ}, we suppose $\boldsymbol{\mu}^\top \boldsymbol{\Sigma} \boldsymbol{\mu} > 0$.*

- We estimate the MCVs or their reciprocals by:

$$\widehat{C}^{\mathrm{RR}} = \sqrt{\frac{(\det \widehat{\boldsymbol{\Sigma}})^{1/d}}{\widehat{\boldsymbol{\mu}}^\top \widehat{\boldsymbol{\mu}}}},\ \widehat{C}^{\mathrm{VV}} = \sqrt{\frac{\mathrm{tr}\widehat{\boldsymbol{\Sigma}}}{\widehat{\boldsymbol{\mu}}^\top \widehat{\boldsymbol{\mu}}}},\ \widehat{C}^{\mathrm{VN}} = \sqrt{\frac{1}{\widehat{\boldsymbol{\mu}}^\top \widehat{\boldsymbol{\Sigma}}^{-1} \widehat{\boldsymbol{\mu}}}},\ \widehat{C}^{\mathrm{AZ}} = \sqrt{\frac{\widehat{\boldsymbol{\mu}}^\top \widehat{\boldsymbol{\Sigma}} \widehat{\boldsymbol{\mu}}}{(\widehat{\boldsymbol{\mu}}^\top \widehat{\boldsymbol{\mu}})^2}}, \widehat{B}^v = 1/\widehat{C}^v,$$

where

$$\widehat{\boldsymbol{\mu}} = \frac{1}{n_1}\sum_{j=1}^{n_1} \boldsymbol{X}_j,\ \widehat{\boldsymbol{\Sigma}} = \frac{1}{n_1}\sum_{j=1}^{n_1} (\boldsymbol{X}_j - \widehat{\boldsymbol{\mu}})(\boldsymbol{X}_j - \widehat{\boldsymbol{\mu}})^\top.$$

Theorem 1

Assume that $n_1 \to \infty$. Let $v \in \{RR, VV, VN, AZ\}$ and Assumption 1 be fulfilled.

1. *The estimators $\widehat{C}^v$ are asymptotically normal,*

$$n_1^{1/2}\left(\widehat{C}^v - C^v\right) \xrightarrow{d} Z_{C^v} \sim N(0, \sigma^2_{C^v}),$$

$$\sigma^2_{C^v} = \frac{S_v}{4}\boldsymbol{A}_v(\boldsymbol{\mu}, \boldsymbol{\Sigma})\begin{pmatrix} \boldsymbol{\Sigma} & \boldsymbol{\Psi}_3^\top \\ \boldsymbol{\Psi}_3 & \boldsymbol{\Psi}_4 \end{pmatrix}\boldsymbol{A}_v(\boldsymbol{\mu}, \boldsymbol{\Sigma})^\top,$$

   where $S_{RR} = d^{-2}(C^{RR})^{2-4d},\ S_{VV} = (C^{VV})^{-2},\ S_{VN} = (C^{VN})^6,\ S_{AZ} = (C^{AZ})^{-2}.$

2. *The reciprocals $\widehat{B}_i^v$ are asymptotically normal as well,*

$$n_1^{1/2}\left(\widehat{B}^v - B^v\right) \xrightarrow{d} Z_{B^v} \sim N(0, \sigma^2_{B^v}) \quad \text{with } \sigma^2_{B^v} = (C^v)^{-4}\sigma^2_{C^v}\,.$$

$$\boldsymbol{A}_{\mathrm{RR}}(\boldsymbol{\mu},\boldsymbol{\Sigma}) = \left(-2d\ \det(\boldsymbol{\Sigma}) \frac{\boldsymbol{\mu}^\top}{(\boldsymbol{\mu}^\top\boldsymbol{\mu})^{d+1}} + \frac{\det(\boldsymbol{\Sigma})\left(\mathrm{vec}(\boldsymbol{\Sigma}^{-1})\right)^\top}{(\boldsymbol{\mu}^\top\boldsymbol{\mu})^d} \widetilde{\boldsymbol{D}}(\boldsymbol{\mu}), \frac{\det(\boldsymbol{\Sigma})\left(\mathrm{vec}(\boldsymbol{\Sigma}^{-1})\right)^\top}{(\boldsymbol{\mu}^\top\boldsymbol{\mu})^d} \right),$$

$$\boldsymbol{A}_{\mathrm{VV}}(\boldsymbol{\mu},\boldsymbol{\Sigma}) = \left(-2\,\mathrm{tr}(\boldsymbol{\Sigma}) \frac{\boldsymbol{\mu}^\top}{(\boldsymbol{\mu}^\top\boldsymbol{\mu})^2} + \frac{1}{\boldsymbol{\mu}^\top\boldsymbol{\mu}} (\mathrm{vec}(\boldsymbol{I}_d))^\top \widetilde{\boldsymbol{D}}(\boldsymbol{\mu}), \frac{1}{\boldsymbol{\mu}^\top\boldsymbol{\mu}} (\mathrm{vec}(\boldsymbol{I}_d))^\top \right),$$

$$\boldsymbol{A}_{\mathrm{VN}}(\boldsymbol{\mu},\boldsymbol{\Sigma}) = \left(2\,\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1} - [(\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}) \otimes (\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1})]\widetilde{\boldsymbol{D}}(\boldsymbol{\mu}), -(\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}) \otimes (\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}) \right),$$

$$\boldsymbol{A}_{\mathrm{AZ}}(\boldsymbol{\mu},\boldsymbol{\Sigma}) = \left(-4\,\boldsymbol{\mu}^\top\boldsymbol{\Sigma}\boldsymbol{\mu} \frac{\boldsymbol{\mu}^\top}{(\boldsymbol{\mu}^\top\boldsymbol{\mu})^3} + 2\frac{\boldsymbol{\mu}^\top\boldsymbol{\Sigma}}{(\boldsymbol{\mu}^\top\boldsymbol{\mu})^2} + \frac{\boldsymbol{\mu}^\top \otimes \boldsymbol{\mu}^\top}{(\boldsymbol{\mu}^\top\boldsymbol{\mu})^2} \widetilde{\boldsymbol{D}}(\boldsymbol{\mu}), \frac{\boldsymbol{\mu}^\top \otimes \boldsymbol{\mu}^\top}{(\boldsymbol{\mu}^\top\boldsymbol{\mu})^2} \right)$$

where $\otimes$ is the Kronecker product, and the matrices $\widetilde{\boldsymbol{D}}(\boldsymbol{x}) \in \mathbb{R}^{d^2 \times d}$ for $\boldsymbol{x} = (x_1, \dots, x_d)^\top \in \mathbb{R}^d$, $\boldsymbol{\Psi}_{i3} \in \mathbb{R}^{d^2 \times d}$ as well as $\boldsymbol{\Psi}_{i4} \in \mathbb{R}^{d^2 \times d^2}$ are given by their entries

$$
\begin{aligned}
[\widetilde{\boldsymbol{D}}(\boldsymbol{x})]_{ad-d+r,s} &= -x_r I\{s = a \neq r\} - 2x_s I\{s = r = a\} - x_a I\{r = s \neq a\} \\
[\boldsymbol{\Psi}_{i3}]_{ad-d+r,s} &= E(X_{i1a}X_{i1r}X_{i1s}) - E(X_{i1a}X_{i1r})E(X_{i1s}) \\
[\boldsymbol{\Psi}_{i4}]_{ad-d+r,bd-d+s} &= E(X_{i1a}X_{i1r}X_{i1b}X_{i1s}) - E(X_{i1a}X_{i1r})E(X_{i1b}X_{i1s})
\end{aligned}
$$

for $a, b, r, s \in \{1, \dots, d\}$.

- The variances $\sigma^2_{B^v}$ and $\sigma^2_{C^v}$ can be naturally estimated by replacing the expectations and covariances by their empirical counterparts, for instance:

$$[\widehat{\boldsymbol{\Psi}}_3]_{ad-d+r,s} = \left(n_1^{-1}\sum_{j=1}^{n_1} X_{ja}X_{jr}X_{js}\right) - \left(n_1^{-1}\sum_{j=1}^{n_1} X_{ja}X_{jr}\right)\left(n_1^{-1}\sum_{j=1}^{n_1} X_{js}\right).$$

- We obtain

$$\widehat{\sigma}^2_{C^v} = \frac{\widehat{S}_v}{4}\boldsymbol{A}_v(\widehat{\boldsymbol{\mu}},\widehat{\boldsymbol{\Sigma}})\begin{pmatrix}\widehat{\boldsymbol{\Sigma}} & \widehat{\boldsymbol{\Psi}}_3^\top \\ \widehat{\boldsymbol{\Psi}}_3 & \widehat{\boldsymbol{\Psi}}_4\end{pmatrix}\boldsymbol{A}_v(\widehat{\boldsymbol{\mu}},\widehat{\boldsymbol{\Sigma}})^\top, \quad \widehat{\sigma}^2_{B^v} = (\widehat{C}^v)^{-4}\widehat{\sigma}^2_{C^v}, \tag{2}$$

where $\widehat{S}_{\mathrm{RR}} = d^{-2}(\widehat{C}^{\mathrm{RR}})^{2-4d}$, $\widehat{S}_{\mathrm{VV}} = (\widehat{C}^{\mathrm{VV}})^{-2}$, $\widehat{S}_{\mathrm{VN}} = (\widehat{C}^{\mathrm{VN}})^{6}$, $\widehat{S}_{\mathrm{AZ}} = (\widehat{C}^{\mathrm{AZ}})^{-2}$.

Lemma 2 (Consistent variance estimators)

Under Assumption 1, $\widehat{\sigma}^2_{C^v} \xrightarrow{p} \sigma^2_{C^v}$ *and* $\widehat{\sigma}^2_{B^v} \xrightarrow{p} \sigma^2_{B^v}$.

Definition 3 (Conditional two-point distribution)

Let $\boldsymbol{Y} = (Y_1, \dots, Y_d)^\top \in \mathbb{R}^d$ be a multivariate random variable. We call the rth coordinate Y_r *conditionally two-point distributed* if it is (conditionally) degenerated or it just takes (conditionally) two different values with positive probability, both given the remaining components $(Y_s)_{s=1,\dots,d; s\neq r}$.

Assumption 2

No coordinate of $\boldsymbol{X}_1$ is conditionally two-point distributed.

Lemma 4 (Nondegenerecy)

Under Assumptions 1 and 2 we have $\sigma^2_{C^v} > 0$ and, thus, $\sigma^2_{B^v} > 0$.

- Let

$$\boldsymbol{X}_{ij} = (X_{ij1}, \dots, X_{ijd})^\top \quad (i = 1, \dots, k; j = 1, \dots, n_i),$$

where $\boldsymbol{X}_{i1}, \dots, \boldsymbol{X}_{in_i}$ are identically distributed for each $i = 1, \dots, k$ and all observations $\boldsymbol{X}_{11}, \dots, \boldsymbol{X}_{kn_k}$ are mutually independent.

- We choose a contrast matrix $\boldsymbol{H} \in \mathbb{R}^{r \times k}$, i.e. $\boldsymbol{H}\mathbf{1}_{k\times 1} = \mathbf{0}_{r\times 1}$. We like to infer:

$$\mathcal{H}_{0,C^v} : \boldsymbol{H}\boldsymbol{C}^v = \mathbf{0}_{r\times 1}, \quad \mathcal{H}_{0,B^v} : \boldsymbol{H}\boldsymbol{B}^v = \mathbf{0}_{r\times 1}, \tag{3}$$

where $\boldsymbol{C}^v = (C_1^v, \dots, C_k^v)^\top$ and $\boldsymbol{B}^v = (B_1^v, \dots, B_k^v)^\top$ $(v = \mathrm{RR}, \mathrm{VV}, \mathrm{VN}, \mathrm{AZ})$.

- In the classical k-sample scenario, the null hypothesis of no group effect can be tested by

$$\mathcal{H}_{0,C^v} : \{\boldsymbol{P}_k\boldsymbol{C}^v = \mathbf{0}_{k\times 1}\} = \{C_1^v = \ldots = C_k^v\},$$

where

$$\boldsymbol{P}_k = \boldsymbol{I}_k - \mathbf{1}_{k\times k}/k.$$

- Let

$$\frac{n_i}{n} \to \kappa_i \in (0,1) \text{ for all } i = 1, \dots, k, \;\; n = \sum_{i=1}^{k} n_i. \tag{4}$$

- The Wald-type statistics for testing general null hypotheses (3):

$$S_{n,C^v}(\boldsymbol{H}) = n(\boldsymbol{H}\widehat{\boldsymbol{C}}^v)^\top(\boldsymbol{H}\widehat{\boldsymbol{\Sigma}}_{C^v}\boldsymbol{H}^\top)^+\boldsymbol{H}\widehat{\boldsymbol{C}}^v, \;\; S_{n,B^v}(\boldsymbol{H}) = n(\boldsymbol{H}\widehat{\boldsymbol{B}}^v)^\top(\boldsymbol{H}\widehat{\boldsymbol{\Sigma}}_{B^v}\boldsymbol{H}^\top)^+\boldsymbol{H}\widehat{\boldsymbol{B}}^v,$$

where

$$\widehat{\boldsymbol{\Sigma}}_{C^v} = \operatorname{diag}((n/n_1)\widehat{\sigma}^2_{1,C^v}, \dots, (n/n_k)\widehat{\sigma}^2_{k,C^v})$$
$$\widehat{\boldsymbol{\Sigma}}_{B^v} = \operatorname{diag}((n/n_1)\widehat{\sigma}^2_{1,B^v}, \dots, (n/n_k)\widehat{\sigma}^2_{k,B^v})$$

are diagonal matrices, and $\boldsymbol{A}^+$ is the Moore–Penrose inverse for a matrix $\boldsymbol{A}$.

Theorem 5

Let (4) as well as Assumptions 1 and 2 be fulfilled for all (sub-)groups $i = 1, \dots, k$.

(i) *Under* $\mathcal{H}_{0,C^v} : \boldsymbol{HC}^v = \mathbf{0}$, $S_{n,C^v}(\boldsymbol{H})$ *tends in distribution to* $Z \sim \chi^2_{rank(\boldsymbol{H})}$.

(ii) *Under* $\mathcal{H}_{1,C^v} : \boldsymbol{HC}^v \neq \mathbf{0}$, $S_{n,C^v}(\boldsymbol{H})$ *diverges, i.e.* $S_{n,C^v}(\boldsymbol{H})$ *converges in probability to* ∞.

(iii) *Under* $\mathcal{H}_{0,B^v} : \boldsymbol{HB}^v = \mathbf{0}$, $S_{n,B^v}(\boldsymbol{H})$ *tends in distribution to* $Z \sim \chi^2_{rank(\boldsymbol{H})}$.

(iv) *Under* $\mathcal{H}_{1,B^v} : \boldsymbol{HB}^v \neq \mathbf{0}$, $S_{n,B^v}(\boldsymbol{H})$ *diverges, i.e.* $S_{n,B^v}(\boldsymbol{H})$ *converges in probability to* ∞.

- We obtain asymptotically valid tests

$$\varphi_{n,C^v} = \mathbf{1}\{S_{n,C^v}(\boldsymbol{H}) > \chi^2_{\mathrm{rank}(\boldsymbol{H}),1-\alpha}\},$$
$$\varphi_{n,B^v} = \mathbf{1}\{S_{n,B^v}(\boldsymbol{H}) > \chi^2_{\mathrm{rank}(\boldsymbol{H}),1-\alpha}\},$$

i.e. they have an asymptotic level α and an asymptotic power of 1.

- To improve finite sample properties of the Wald-type statistic, we consider permutation and bootstrap tests.
- We first group all data together resulting in pooled data by $\boldsymbol{X} = (\boldsymbol{X}_{ij})_{i=1,\dots,k;j=1,\dots,n_i}$.
- We draw without or with replacement from $\boldsymbol{X}$ to obtain
  - a permutation $\boldsymbol{X}^{\pi} = (\boldsymbol{X}_{ij}^{\pi})_{i=1,\dots,k;j=1,\dots,n_i}$,
  - a bootstrap $\boldsymbol{X}^{b} = (\boldsymbol{X}_{ij}^{b})_{i=1,\dots,k;j=1,\dots,n_i}$

  sample, respectively.
- The benefit of the permutation approach is its finite exactness under exchangeability, here under $\widetilde{\mathcal{H}}_0 : \boldsymbol{X}_{11} \overset{d}{=} \dots \overset{d}{=} \boldsymbol{X}_{k1}$.

- The expectation $\boldsymbol{\mu}_0 = \sum_{i=1}^k \kappa_i \boldsymbol{\mu}_i$ and the covariance matrix $\boldsymbol{\Sigma}_0$ for the (asymptotic) pooled distribution $P_0 = \sum_{i=1}^k \kappa_i P^{\boldsymbol{X}_{i1}}$, where $[\boldsymbol{\Sigma}_0]_{\ell m} = (\sum_{i=1}^k \kappa_i E(X_{\ell 1} X_{m1})) - [\boldsymbol{\mu}_0]_\ell [\boldsymbol{\mu}_0]_m$.
- Assumption 2 is true for the pooled distribution when this is the case for all (sub-)groups.

Theorem 6

In addition to the assumptions of Theorem 1, we suppose that Assumption 1 is fulfilled for $\boldsymbol{\mu}_0$ *and* $\boldsymbol{\Sigma}_0$*. Then the following statements are valid independently whether the respective null hypotheses* $\mathcal{H}_{0,C^v}$, $\mathcal{H}_{0,B^v}$ *or their respective alternatives are true:*

a. *The permutation statistics* $S^\pi_{n,C^v}(\boldsymbol{H})$ *and* $S^\pi_{n,B^v}(\boldsymbol{H})$ *always mimic the null distribution limit of* $S_{n,C^v(\boldsymbol{H})}$ *and* $S_{n,B^v}(\boldsymbol{H})$ *asymptotically, respectively, i.e.,*

$$\sup_{x \in \mathbb{R}} \left| \Pr\left(S^\pi_{n,C^v}(\boldsymbol{H}) \le x \mid \boldsymbol{X} \right) - \chi^2_{rank(\boldsymbol{H})}(x) \right| \xrightarrow{p} 0.$$

b. *The bootstrap statistics* $S^b_{n,C^v}(\boldsymbol{H})$ *and* $S^b_{n,B^v}(\boldsymbol{H})$ *always mimic the null distribution limit of* $S_{n,C^v(\boldsymbol{H})}$ *and* $S_{n,B^v}(\boldsymbol{H})$ *asymptotically, respectively.*

- The multiple testing problem

$$\mathcal{H}_{0,\ell,\boldsymbol{C}^v} : \boldsymbol{h}_\ell^\top \boldsymbol{C}^v = 0 \qquad (\ell = 1, \ldots, r) \tag{5}$$

for contrast vectors $\boldsymbol{h}_\ell \in \mathbb{R}^k$, i.e. $\boldsymbol{h}_\ell^\top \mathbf{1}_{k\times 1} = 0$.

- The intersection

$$\bigcap_{\ell=1}^{r} \mathcal{H}_{0,\ell,\boldsymbol{C}^v}$$

of the local null hypotheses coincides with the global null hypothesis $\mathcal{H}_{0,\boldsymbol{C}^v}$ from (3) with $\boldsymbol{H} = (\boldsymbol{h}_1, \ldots, \boldsymbol{h}_r)^\top$.

- In the easiest case, we are interested in group differences, i.e. the global null hypotheses is $\mathcal{H}_{0,C^v} : C_1^v = \ldots = C_k^v$.
- Tukey's all-pairs comparison (Tukey, 1953)

$$\mathcal{H}_{0,C^v} : C_1^v = \ldots = C_k^v \Leftrightarrow$$

$$\mathcal{H}_{0,C^v} : \begin{cases} C_1^v = C_2^v \\ C_1^v = C_3^v \\ \vdots \\ C_1^v = C_k^v \\ C_2^v = C_3^v \\ \vdots \\ C_{k-1}^v = C_k^v \end{cases} \Leftrightarrow \mathcal{H}_{0,C^v} : \begin{pmatrix} -1 & 1 & 0 & \ldots & \ldots & 0 & 0 \\ -1 & 0 & 1 & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \ldots & \ldots & 1 \\ 0 & -1 & 1 & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 & -1 & 1 \end{pmatrix} \boldsymbol{C}^v = \boldsymbol{0}$$

- We consider the following max-type statistic

$$S_{n,\max,C^v}(\boldsymbol{H}) = \max_{\ell=1,\ldots,r} |T^v_{\ell,n}|, \qquad T^v_{\ell,n} = \sqrt{n}\frac{\boldsymbol{h}_\ell^\top \widehat{\boldsymbol{C}}^v}{\sqrt{\boldsymbol{h}_\ell^\top \widehat{\boldsymbol{\Sigma}}_{C^v} \boldsymbol{h}_\ell}},$$

  where $|T^v_{\ell,n}|$ equals $(S_{n,C^v}(\boldsymbol{h}_\ell^\top))^{1/2}$.
- By Theorem 1, $(T^v_{1,n}, \ldots, T^v_{r,n})$ converges in distribution to a multivariate normal distribution with standard normal distributed marginals and correlation matrix $\boldsymbol{R}_{C^v}$ given by its entries

$$[\boldsymbol{R}_{C^v}]_{\ell m} = \frac{\boldsymbol{h}_\ell^\top \boldsymbol{\Sigma}_{C^v} \boldsymbol{h}_m}{\sqrt{\boldsymbol{h}_\ell^\top \boldsymbol{\Sigma}_{C^v} \boldsymbol{h}_\ell}\sqrt{\boldsymbol{h}_m^\top \boldsymbol{\Sigma}_{C^v} \boldsymbol{h}_m}}. \tag{6}$$

- The equicoordinate $(1-\alpha)$-quantile $q_{1-\alpha,\max,C^v}(\boldsymbol{R})$ of a $N(\mathbf{0}, \boldsymbol{R}_{C^v})$-distribution serves as a "fair" critical value.
- Such quantiles can be determined numerically by computer software, e.g. the function *qmvnorm()* from the R-package *mvtnorm* (R Core Team, 2023; Genz et al., 2021; Genz and Bretz, 2009).
- An asymptotically exact test

$$\varphi_{n,\max,C^v} = \mathbf{1}\{S_{n,\max,C^v}(\boldsymbol{H}) > q_{1-\alpha,\max,C^v}(\widehat{\boldsymbol{R}}_{C^v})\}$$

  for the global null hypothesis $\mathcal{H}_{0,C^v}$.
- We reject the local null hypothesis $\mathcal{H}_{0,\ell,C^v}$ when $T^v_{\ell,n} > q_{1-\alpha,\max,C^v}(\widehat{\boldsymbol{R}}_{C^v})$.

Theorem 7

Let (4) as well as Assumptions 1 and 2 be fulfilled for all (sub-)groups $i = 1, \ldots, k$*.*

- a *The test* φ_{n,max,C^v} *is asymptotically exact for the global null* $\mathcal{H}_{0,C^v}$*, i.e.* $E_{\mathcal{H}_{0,C^v}}(\varphi_{n,max,C^v}) \to \alpha$*.*
- b *Suppose that the first* $r' \leq r$ *null hypotheses and the remaining* $r - r'$ *alternatives, i.e.* $\mathcal{H}_{1,\ell,C^v} : \boldsymbol{h}_\ell^\top \boldsymbol{C}^v \neq 0$ *for* $\ell = r' + 1, \ldots, r$*, are true. Then*

$$\limsup_{n\to\infty} \Pr\Big(\bigcup_{\ell=1}^{r'} \{|T_{\ell,n}^v| > q_{1-\alpha,max,C^v}(\widehat{\boldsymbol{R}}_{C^v})\}\Big) \leq \alpha$$

$$\text{and} \quad \lim_{n\to\infty} \Pr\Big(\bigcap_{\ell=r'+1}^{r} \{|T_{\ell,n}^v| > q_{1-\alpha,max,C^v}(\widehat{\boldsymbol{R}}_{C^v})\}\Big) = 1.$$

- c *The statements remain true when* $\boldsymbol{C}^v$ *is replaced by* $\boldsymbol{B}^v$*.*

- Using the pooled bootstrap, we propose to approximate $n^{1/2}(\widehat{\boldsymbol{C}}^v - \boldsymbol{C}^v)$ by

$$n^{1/2}\widehat{\boldsymbol{\Sigma}}_{C^v}^{1/2}(\widehat{\boldsymbol{\Sigma}}_{C^v}^{b})^{-1/2}(\widehat{\boldsymbol{C}}^{vb} - \widehat{\boldsymbol{C}}_0^v), \qquad \widehat{\boldsymbol{C}}_0^v = \widehat{C}_0^v \cdot \mathbf{1}_{k\times 1}.$$

- The bootstrap multiple contrast statistic becomes

$$S_{n,\max,C^v}^b(\boldsymbol{H}) = n^{1/2} \max_{\ell=1,\dots,r} |T_{\ell,n}^{v,b}|, \qquad T_{\ell,n}^{v,b} = \frac{\boldsymbol{h}_\ell^\top \widehat{\boldsymbol{\Sigma}}_{C^v}^{1/2}(\widehat{\boldsymbol{\Sigma}}_{C^v}^{b})^{-1/2}(\widehat{\boldsymbol{C}}^{vb} - \widehat{\boldsymbol{C}}_0^v)}{\sqrt{\boldsymbol{h}_\ell^\top \widehat{\boldsymbol{\Sigma}}_{C^v} \boldsymbol{h}_\ell}}.$$

- Then

$$\varphi_{n,\max,C^v}^b = \mathbf{1}\{S_{n,\max,C^v}(\boldsymbol{H}) > q_{1-\alpha,\max,C^v}^b(\boldsymbol{X})\}$$

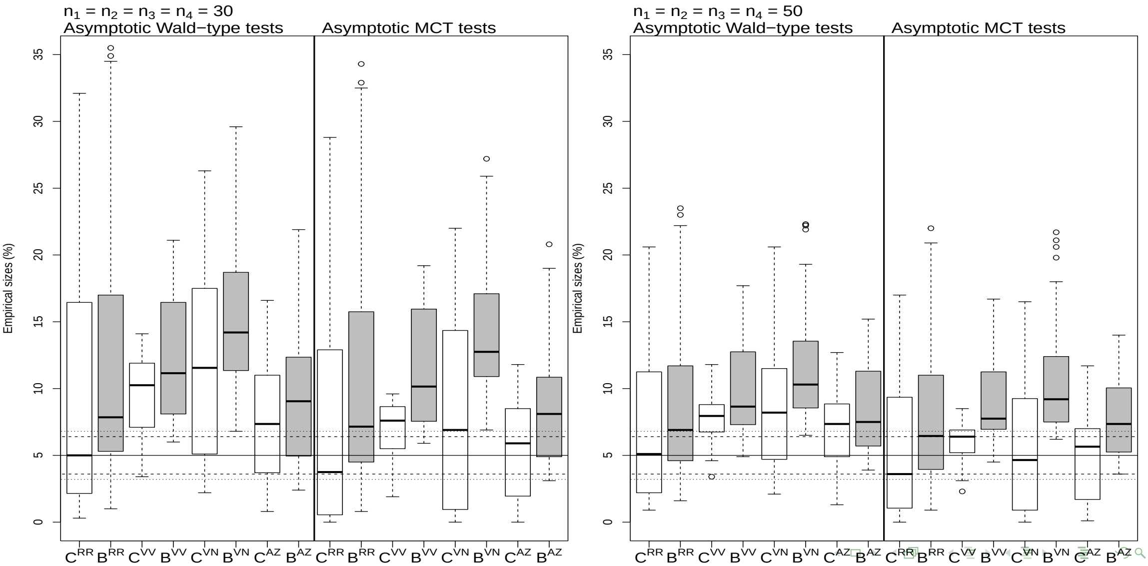
is the bootstrap counterpart of the multiple contrast test, where $q_{1-\alpha,\max,C^v}^b(\boldsymbol{X})$ is the conditional, equicoordinate $(1-\alpha)$-quantile of $n^{1/2}(T_{1,n}^{v,b},\dots,T_{r,n}^{v,b})^\top$ given the data $\boldsymbol{X}$.

Theorem 8

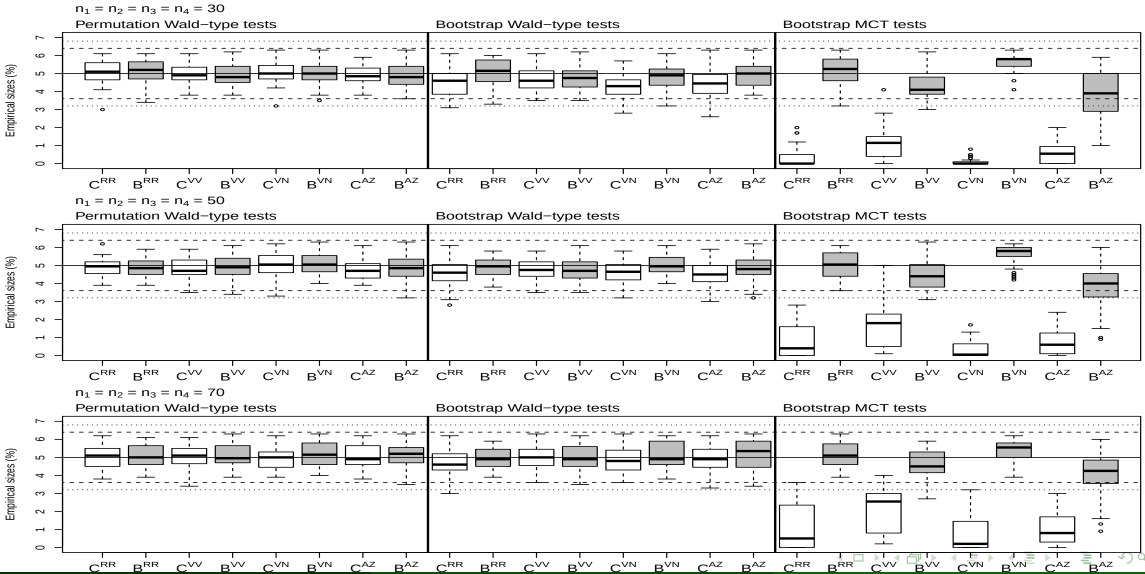
In addition to the assumptions of Theorem 1, we suppose that Assumption 1 is fulfilled for the pooled quantities $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$. Then the statements of Theorem 7 remain true when we replace φ_{n,max,C^v} and $q_{1-\alpha,max,C^v}(\widehat{\boldsymbol{R}}_{C^v})$ by their bootstrap counterparts φ^b_{n,max,C^v} and $q^b_{1-\alpha,max,C^v}(\boldsymbol{X})$, respectively. Moreover, the analogue results for B instead of C are true.

- The asymptotic covariance structure of $n^{1/2}(\widehat{\boldsymbol{C}}^{v\pi} - \widehat{\boldsymbol{C}}_0^{v})$ is more complicated than the bootstrap one, which is caused by the strong dependence within the permutation sample. In particular, the permutation covariance matrix is neither diagonal nor regular.

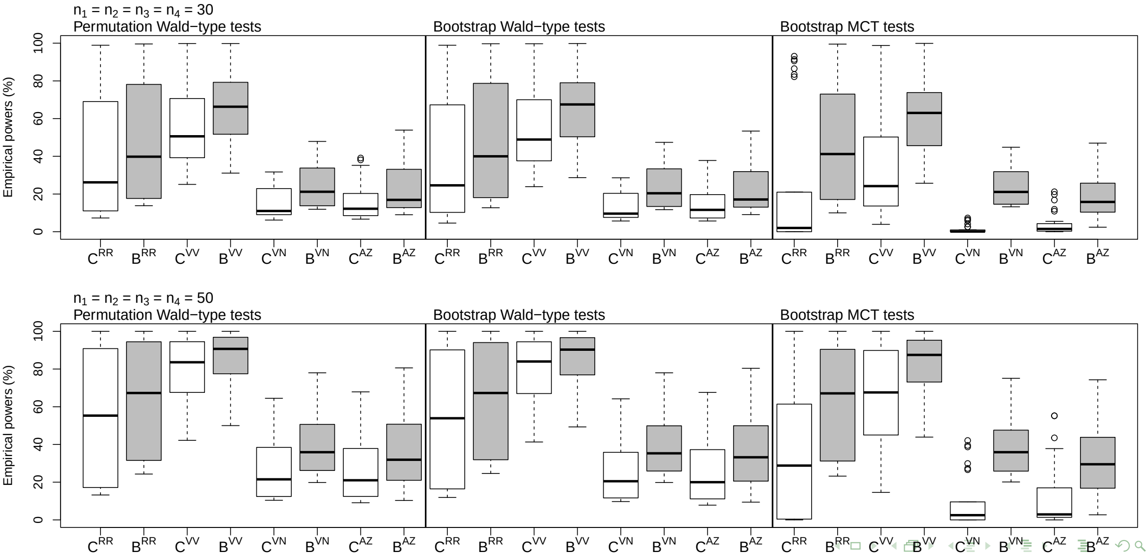
Type-1 error control



Type-1 error control



Empirical power



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-
```
# contrast matrices
k <- length(data_set)
# Tukey's contrast matrix
h_mct <- contr_mat(k, type = "Tukey")
# centering matrix P_k
h_wald <- contr_mat(k, type = "center")
# testing without parallel computing
res <- GFDmcv(data_set, h_mct, h_wald)
summary(res, digits = 3)
```

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Thank you for your attention!