ROTATION SAMPLING
AND
CHEBYSHEV POLYNOMIALS

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Plan

1. Rotation sampling
2. General solution: ASSUMPTIONS I and II
3. ASSUMPTION I and Chebyshev polynomials

Patterson model

	$\cdots$	j	$j+1$	$j+2$	$\cdots$	occasions
	$\cdots$	μ_j	μ_{j+1}	μ_{j+2}	$\cdots$	means
		1	ρ	ρ^2	$\cdots$	correlations $\mathbb{C}\text{orr}(X_{i,j},\ X_{i,j+k})$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
j	$\cdots$	$X_{j,j}$	$X_{j,j+1}$	$X_{j,j+2}$	$\cdots$	
$j+1$	$\cdots$	$X_{j+1,j}$	$X_{j+1,j+1}$	$X_{j+1,j+2}$	$\cdots$	
$j+2$	$\cdots$	$X_{j+2,j}$	$X_{j+2,j+1}$	$X_{j+2,j+2}$	$\cdots$	
$j+3$	$\cdots$	$X_{j+3,j}$	$X_{j+3,j+1}$	$X_{j+3,j+2}$	$\cdots$	
$j+4$	$\cdots$	$X_{j+4,j}$	$X_{j+4,j+1}$	$X_{j+4,j+2}$	$\cdots$	
$j+5$	$\cdots$	$X_{j+5,j}$	$X_{j+5,j+1}$	$X_{j+5,j+2}$	$\cdots$	
$j+6$	$\cdots$	$X_{j+6,j}$	$X_{j+6,j+1}$	$X_{j+6,j+2}$	$\cdots$	
$j+7$	$\cdots$	$X_{j+7,j}$	$X_{j+7,j+1}$	$X_{j+7,j+2}$	$\cdots$	
$j+8$	$\cdots$	$X_{j+8,j}$	$X_{j+8,j+1}$	$X_{j+8,j+2}$	$\cdots$	
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
units						

Patterson model, continuation

Let $X_{i,j}$ be the (random) value of the variable for the ith unit on jth occasion, $i \in \mathbb{N} = \{1, 2, \ldots\}$, $j \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.
For every $j \in \mathbb{Z}$ let

$$\mu_j = \mathbb{E}\, X_{i,j}$$

does not depend on $i \in \mathbb{N}$.

For simplicity assume $\mathbb{V}\mathrm{ar}\, X_{i,j} \equiv 1$.

Moreover, assume that for $0 < |\rho| < 1$

$$\mathbb{C}\mathrm{orr}(X_{i,j}, X_{k,\ell}) = \rho^{|j-\ell|}\, \delta_{i=k}. \qquad (1)$$

Patterson model, continuation

For $N \geq 2$ let $\underline{X}_j = (X_{j,j}, \ldots, X_{j+N-1,j})$, denote the "maximal" sample on the occasion $j \in \mathbb{Z}$.

Then $N \times N$ matrix $\mathbf{C} = \mathbb{C}\mathrm{ov}\,(\underline{X}_j, \underline{X}_{j+1})$ has all entries equal zero except the ones just above the diagonal, which are all equal ρ.

Moreover, (1) yields

$$\mathbb{C}\mathrm{ov}(\underline{X}_j, \underline{X}_k) = \mathbf{C}^{|k-j|}$$

and $\mathbf{C}^r$ is a zero matrix for $r \geq N$ and for $r < N$ it has the rth over-diagonal with all entries equal ρ^r and all other entries are zeros (i.e. $\mathbf{C}$ is a nilpotent matrix of degree N)

The BLUE and the rotation pattern

The BLUE of μ_t is then

$$\hat{\mu}_t = \sum_{s \le t} \sum_{j=1}^{N} \omega_{s,j}(t)\, X_{s+j-1,s},$$

with coefficients $(\omega_{s,j}(t))$ minimizing the variance of $\hat{\mu}_t$ under the unbiasedness constraint

$$\sum_{j=1}^{N} \omega_{s,j}(t) = 0, \quad s < t \quad \text{and} \quad \sum_{j=1}^{N} \omega_{t,j}(t) = 1.$$

A rotation pattern

$$(\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}, \epsilon_N)^T \in \{0, 1\}^N$$

is such that $\varepsilon_1 = \varepsilon_N = 1$. Let $M = \{i : \varepsilon_i = 0\}$.
Then the sample on jth occasion is not $\underline{X}_j$ but

$$(X_{j+k-1,j})_{k \in M^c}, \quad j \in \mathbb{Z}.$$

Let's see it for the pattern **1101001**

	$\cdots$	j	$j+1$	$j+2$	$\cdots$	occasions
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
j	$\cdots$	**1**			$\cdots$	
$j+1$	$\cdots$	**1**	**1**		$\cdots$	
$j+2$	$\cdots$	**0**	**1**	**1**	$\cdots$	
$j+3$	$\cdots$	**1**	**0**	**1**	$\cdots$	
$j+4$	$\cdots$	**0**	**1**	**0**	$\cdots$	
$j+5$	$\cdots$	**0**	**0**	**1**	$\cdots$	
$j+6$	$\cdots$	**1**	**0**	**0**	$\cdots$	
$j+7$	$\cdots$		**1**	**0**	$\cdots$	
$j+8$	$\cdots$			**1**	$\cdots$	
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
units						

Let's see it for the pattern **1101001**

	⋯		j		$j+1$		$j+2$	⋯	occasions
⋮	⋱		⋮		⋮		⋮	⋱	
j	⋯	**1**	$\mathbf{X_{j,j}}$		$X_{j,j+1}$		$X_{j,j+2}$	⋯	
$j+1$	⋯	**1**	$\mathbf{X_{j+1,j}}$	**1**	$\mathbf{X_{j+1,j+1}}$		$X_{j+1,j+2}$	⋯	
$j+2$	⋯	**0**	$X_{j+2,j}$	**1**	$\mathbf{X_{j+2,j+1}}$	**1**	$\mathbf{X_{j+2,j+2}}$	⋯	
$j+3$	⋯	**1**	$\mathbf{X_{j+3,j}}$	**0**	$X_{j+3,j+1}$	**1**	$\mathbf{X_{j+3,j+2}}$	⋯	
$j+4$	⋯	**0**	$X_{j+4,j}$	**1**	$\mathbf{X_{j+4,j+1}}$	**0**	$X_{j+4,j+2}$	⋯	
$j+5$	⋯	**0**	$X_{j+5,j}$	**0**	$X_{j+5,j+1}$	**1**	$\mathbf{X_{j+5,j+2}}$	⋯	
$j+6$	⋯	**1**	$\mathbf{X_{j+6,j}}$	**0**	$X_{j+6,j+1}$	**0**	$X_{j+6,j+2}$	⋯	
$j+7$	⋯		$X_{j+7,j}$	**1**	$\mathbf{X_{j+7,j+1}}$	**0**	$X_{j+7,j+2}$	⋯	
$j+8$	⋯		$X_{j+8,j}$		$X_{j+8,j+1}$	**1**	$\mathbf{X_{j+8,j+2}}$	⋯	
⋮	⋱	⋮		⋮		⋮		⋱	
units									

The BLUE under the rotation pattern and stationarity

When t increases the coeffcients $\omega_{s,i}(t)$, $s \leq t$ quickly stop to depend on t, i.e. it is stationary. Therefore the BLUE

$$\hat{\mu}_t = \sum_{-\infty < s \leq t} \sum_{i=1}^{N} w_{s,i} X_{s+i-1,s}$$

has coeffcients minimizing $\mathbb{V}\mathrm{ar}\, \hat{\mu}_t$ under the **unbiasedness** constraints

$$\sum_{i=1}^{N} w_{s,i} = 0 \quad s < t \quad \text{and} \quad \sum_{i=1}^{N} w_{t,i} = 1$$

and the **rotation pattern** constraints (restricting the set of observed variables)

$$w_{s,i}(1 - \epsilon_i) = 0, \quad i = 1, \ldots, N, \quad s \leq t.$$

Recursion for the BLUEs

Since Patterson (1950) it has been postulated that there exist $p \in \mathbb{N}$, $a_1, \ldots, a_p \in \mathbb{R}$ and $\underline{r}_0, \underline{r}_1, \ldots, \underline{r}_p \in \mathbb{R}^N$ such that for any $t \in \mathbb{Z}$ the following recursion formula holds

$$\hat{\mu}_t = a_1 \hat{\mu}_{t-1} + \ldots a_p \hat{\mu}_{t-p} + \underline{r}_0^T \underline{X}_t + \underline{r}_1^T \underline{X}_{t-1} + \ldots + \underline{r}_p^T \underline{X}_{t-p}, \quad (2)$$

where $\underline{a}^T \underline{b} = \sum_{i=1}^N a_i b_i$.

Patterson (1950) proved that (2) holds true in case $\varepsilon_1 = \ldots = \varepsilon_N = 1$ with $p = 1$ and he identified a_1, $\underline{r}_0$ and $\underline{r}_1$.

Kowalski (2009) solved the recursion problem rotation patterns with "holes" of size 1 with $p = 2$.

A. Szarkowski in early 2000's conjectured that for the rotation pattern of the LFS, 110011, the recursion depth is $p = 3$. This was confirmed and fully solved in Wesolowski (2010).

Towards ASSUMPTION I

- Let s be the number of "holes" in the rotation pattern, $m_1, \ldots, m_s$ be the sizes of consequtive "holes" and

$$q := 1 + \max_{1 \le i \le s} m_i.$$

- Let

$$\mathbf{T}_m = \begin{bmatrix} T_0 & T_1 & T_2 & \ldots & T_{m-2} & T_{m-1} \\ T_1 & T_0 & T_1 & \ldots & T_{m-3} & T_{m-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \\ T_{m-2} & T_{m-3} & T_{m-4} & \ldots & T_0 & T_1 \\ T_{m-1} & T_{m-2} & T_{m-3} & \ldots & T_1 & T_0 \end{bmatrix},$$

where T_k is the kth Chebyshev polynomial of the first kind defined by

$$T_k(x) = \cos(k \arccos(x)), k = 0, 1, \ldots$$

Towards ASSUMPTION I, continuation

- Let $\mathbf{R}_m$ be an $m \times m$ invertible constant three-diagonal matrix

$$\mathbf{R}_m = \begin{bmatrix} 1+\rho^2 & -\rho & 0 & \dots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \dots & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & 0 & \dots & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \dots & -\rho & 1+\rho^2 \end{bmatrix}$$

- For $v_\rho(x) = 1+\rho^2 - 2\rho x$ define a polynomial of degree q

$$Q_q(x) = 1-\rho^2+(N-1)v_\rho(x)-v_\rho^2(x)\sum_{i=1}^{s} \mathrm{tr}\left(\mathbf{T}_{m_i}(x)\mathbf{R}_{m_i}^{-1}(\rho)\right).$$

ASSUMPTION I: **Roots of Q_q are distinct and do not belong to $[-1,1]$.**

Towards ASSUMPTION II

- bi-diagonal matrix:

$$\widetilde{\mathbf{H}}_m(d) = \begin{bmatrix} 1 & -\rho d & & \\ & \ddots & \ddots & \\ & & \ddots & -\rho d \\ & & & 1 \end{bmatrix}$$

- tri-diagonal matrix:

$$\mathbf{H}_m(d) = \begin{bmatrix} 1+\rho^2 & -\rho d & & \\ -\rho/d & \ddots & \ddots & \\ & \ddots & \ddots & -\rho d \\ & & -\rho/d & 1+\rho^2 \end{bmatrix}$$

Towards ASSUMPTION II, continuation

- Let $h = m_1 + \ldots + m_s$.
- Let $\widetilde{\mathbf{G}}(d)$ be an $(h+1) \times (h+1)$

$$\widetilde{\mathbf{G}}(d) = \frac{1}{1-\rho^2} \begin{bmatrix} (N-1)(1-\rho d) + 1 - \rho^2 & (1-\rho d)\underline{1}_h^T \\ (1-\rho d)\underline{1}_h & \mathrm{diag}\left(\widetilde{\mathbf{H}}_{m_1}(d), \ldots, \widetilde{\mathbf{H}}_{m_s}(d)\right) \end{bmatrix}$$

- Let $\mathbf{G}(d)$ be an $h \times (h+1)$ matrix

$$\mathbf{G}(d) = \frac{1}{1-\rho^2} \left[(1-\rho/d)(d-\rho)\underline{1}_h,\ \mathrm{diag}\left(\mathbf{H}_{m_1}(d), \ldots, \mathbf{H}_{m_s}(d)\right) \right].$$

Towards ASSUMPTION II, continuation

- If $x \in \mathbb{C}$ and $\Re(x) \notin [-1, 1]$ then the equation $d + 1/d = 2x$ has two roots $d_{\pm}(x)$ such that $|d_-| < 1$ and $|d_+| > 1$.
- For $x_1, \ldots, x_q$ roots of Q_q (see ASSUMPTION I) define $d_i = d_-(x_i)$, $i = 1, \ldots, q$.
- Let $\mathbf{S}$ be a $(qh + h + 1) \times (q(h + 1))$ matrix

$$\mathbf{S} = \begin{bmatrix} \widetilde{\mathbf{G}}(d_1) & \widetilde{\mathbf{G}}(d_2) & \cdots & \widetilde{\mathbf{G}}(d_q) \\ \mathbf{G}(d_1) & 0 & \cdots & 0 \\ 0 & \mathbf{G}(d_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{G}(d_q) \end{bmatrix}.$$

ASSUMPTION II: **Matrix S is of full rank.**

Kowalski & JW (2015), *Survey Methodology*

Theorem (1)

If ASSUMPTION I and ASSUMPTION II are satisfied then

$$\hat{\mu}_t = a_1 \hat{\mu}_{t-1} + \ldots a_p \hat{\mu}_{t-p} + \underline{r}_0^T \underline{X}_t + \underline{r}_1^T \underline{X}_{t-1} + \ldots + \underline{r}_p^T \underline{X}_{t-\nu}$$

with $p = 1 + \max_{1 \leq i \leq s} m_i$,

$$a_k = (-1)^{k+1} \sum_{1 \leq j_1 < \ldots < j_k \leq p} d_{j_1} \ldots d_{j_k}, \quad k = 1, \ldots, p,$$

and the vector coefficients $\underline{r}_i$, $i = 0, 1, \ldots, p$, *explicitly given in terms of the covariance matrix* $\mathbf{C}$, *roots* $d_1, \ldots, d_p$ *and solutions of the linear system*

$$\mathbf{S}\underline{c} = (1, 0, \ldots, 0) \in \mathbb{R}^{ph+h+1}.$$

(see Theorem 3.1 in KW)

Example: Szarkowski's scheme 110011

Then $q = 3$ and

$$Q_q(x) = 1 - \rho^2 + 5v_\rho(x) - 2v_\rho^2(x)\frac{1+\rho^2+x}{1+\rho^2+\rho^4}.$$

For $\rho = 0.7$ the roots of Q_3 are

$$x_{1,2} = -0.5668 \pm 1.4069\,i, \quad x_3 = 1.1336.$$

Then

$$d_{1,2} = -0.0968 \pm 0.2899i, \quad d_3 = 0.5997$$

Example: Szarkowski's scheme 110011, cont.

ASSUMPTIONS I and II are satisfied. By Theorem (1) we get

$$\hat{\mu}_t = 0.4060\,\mu_{t-1} + 0.0227\,\mu_{t-2} + 0.0560\mu_{t-3}$$

$$+ \begin{bmatrix} 0.2862 \\ 0.2217 \\ 0.0000 \\ 0.0000 \\ 0.2862 \\ 0.2059 \end{bmatrix}^T \underline{X}_t + \begin{bmatrix} -0.0036 \\ -0.2004 \\ 0.0000 \\ 0.0000 \\ -0.0036 \\ 0.1984 \end{bmatrix}^T \underline{X}_{t-1}$$

$$+ \begin{bmatrix} -0.0143 \\ -0.0026 \\ 0.0000 \\ 0.0000 \\ -0.0143 \\ 0.0033 \end{bmatrix}^T \underline{X}_{t-2} + \begin{bmatrix} 0.0000 \\ -0.0010 \\ 0.0000 \\ 0.0000 \\ -0.0760 \\ 0.0100 \end{bmatrix}^T \underline{X}_{t-3}$$

Conjecture

Intensive numerical simulations in KW led to the CONJECTURE:

ASSUMPTIONs I and II are universally satisfied.

Partial answer:

For rotation patterns with single "hole" of arbitrary size ASSUMPTION I is satisfied:

Theorem

Polynomial Q_p has

- *exactly one real root when p is odd;*
- *exactly two real roots (when p is even);*

(i.e. all the remaining roots of Q_p are complex).

The real roots of Q_p are outside of interval $[-1, 1]$.

All roots of Q_p are simple.

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The real roots of Q_p are outside of interval $[-1, 1]$.

All roots of Q_p are simple.

Chebyshev polynomials, part 1

Chebyshev polynomials of the first kind $(T_n)_{n\geq 0}$ are defined by $T_0(x) = 1$, $T_1(x) = x$ and

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, \ldots$$

with $\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = c_m\delta_{m=n}$ (orthogonality).

Chebyshev polynomials of the second kind $(U_n)_{n\geq 0}$ are defined by $U_0(x) = 1$, $U_1(x) = 2x$ and

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad n = 1, 2, \ldots$$

with $\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\,dx = d_m\delta_{m=n}$ (orthogonality).

Chebyshev polynomials, part 2

Representations: for $n \geq 0$

$$T_n(\cos t) = \cos(nt) \quad \text{and} \quad U_n(\cos t) = \frac{\sin((n+1)t)}{\sin t};$$

$$T_n\left(\frac{x+\frac{1}{x}}{2}\right) = \frac{x^n + \frac{1}{x^n}}{2} \quad \text{and} \quad U_n\left(\frac{x+\frac{1}{x}}{2}\right) = \frac{x^{n+1} - \frac{1}{x^{n+1}}}{x - \frac{1}{x}}.$$

Basic relations: for $n \geq 0$

$$T_n' = nU_{n-1} \quad \text{and} \quad T_n^2(x) + (1 - x^2)U_{n-1}^2(x) = 1 \tag{3}$$

$$\sum_{j=1}^{n} T_j\, U_{n-j} = \frac{n}{2}\, U_n \tag{4}$$

$$U_n(y) + 2\sum_{j=1}^{n} T_j(x)\, U_{n-j}(y) = \frac{T_{n+1}(x) - T_{n+1}(y)}{x - y}. \tag{5}$$

Chebyshev polynomials, part 3

For all $x \geq 1$ the $n \times n$ matrix $\mathbf{V}_n(x)$ defined by

$$\mathbf{V}_n(x) = \begin{bmatrix} 2x & -1 & 0 & \dots & 0 & 0 \\ -1 & 2x & -1 & \dots & 0 & 0 \\ 0 & -1 & 2x & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & 0 & \dots & 2x & -1 \\ 0 & 0 & 0 & \dots & -1 & 2x \end{bmatrix}$$

is invertible.

Its inverse, $\mathbf{A} = [a_{i,j}]_{i,j \in \{1,\dots,n\}} = \mathbf{V}_n^{-1}(x)$, is symmetric and

$$a_{i,j} = \frac{1}{U_n(x)} U_{i-1}(x) U_{n-j}(x), \quad 1 \leq i \leq j \leq n. \tag{6}$$

Proof (sketch)

In case $s = 1$ and $m_1 = m$ we have $p = m + 1$ and with $v_\rho(x) = 1 + \rho^2 - 2\rho x$

$$Q_p(x) = 1 - \rho^2 + (N-1)v_\rho(x) - v_\rho^2(x)\mathrm{tr}\left(\mathbf{T}_m(x)\mathbf{R}_m^{-1}(\rho)\right).$$

Using (4), (5) and (6) we first show

Lemma

- $$\det \mathbf{R}_m(\rho) = \rho^m U_m(r), \quad m = 0, 1, \ldots,$$
  where $r = (\rho + 1/\rho)/2$;
- $$v_\rho^2(x)\mathrm{tr}\left(\mathbf{T}_m(x)\mathbf{R}_m^{-1}(\rho)\right) = 1-\rho^2+(m+1)v_\rho(x)-2\frac{1-\rho^{m+1}T_{m+1}(x)}{\det\mathbf{R}_m(\rho)}.$$

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Proof (sketch): additive affine perturbation of T_{m+1}

Lemma implies that the roots of Q_p are the same as roots of

$$\widetilde{Q}_p(x) := a + bx + T_{m+1}(x),$$

where

$$a = -(n-2)rU_m(r) - \rho^{-m-1} \quad \text{and} \quad b = (n-2)U_m(r)$$

with $n = N - m$ and $r = (\rho + 1/\rho)/2 \notin [-1, 1]$.

Proof (sketch): Roots of $\widetilde{Q}_p$

In view of (3) the derivative of $\widetilde{Q}_p$ is

$$\widetilde{Q}_p' = b + (m+1)U_m.$$

(i) no multiple complex roots;

(ii) monotonicity of $\widetilde{Q}_p$;

- (m even, $\rho > 0$) $\Rightarrow$ ($a < -1$, $b \geq 0$): one real simple root $x_1 > 1$;
- (m even, $\rho < 0$) $\Rightarrow$ ($a > 1$, $b \geq 0$): one real simple root $x_1 < -1$;
- (m odd, $\rho > 0$) $\Rightarrow$ ($a < -1$, $b \geq 0$): two real simple roots $x_1 < -1$ and $x_2 > 1$;
- (m odd, $\rho < 0$) $\Rightarrow$ ($a < -1$, $b \leq 0$): two real simple roots $x_1 < -1$ and $x_2 > 1$.

Questions

- What about ASSUMPTION II in case of a single "hole" of an arbitrary size?
- What about ASSUMPTIONS I and II in (more) general case(s).
- What about the model with arbitrary correlation (not necessarily exponential in time)?
  (Szarkowski's numerical experiments suggest that then $p = N + 1$. Check it, at least for no "holes".)

Literature

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