Detection and localization of changes in panels of random densities

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Motivation: Brazilian Covid data

Data:

- **What?** Infection data Ct-values
- **Where?** Brazil resolution down to cities
- **When?** 05/15/2020 – 05/15/2022 daily, $\sim$ 105 weeks

Aim:

- Identify changes in space and time
- Changes in viral loads

Data (i)

Typical studies: Counts of cases or deaths

Data (II)

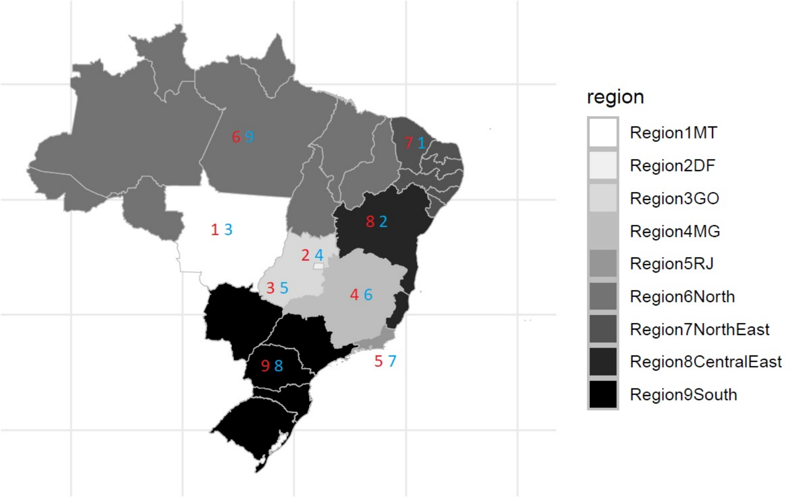
Ct-values:

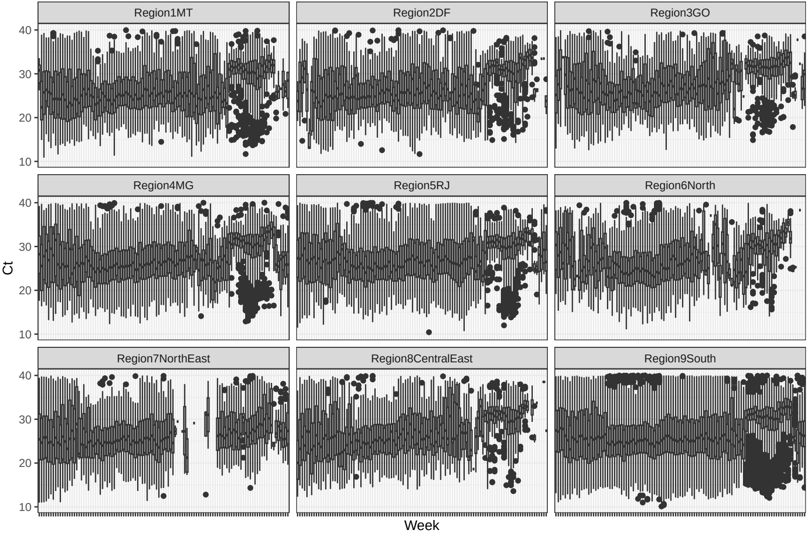
- Measured by PCR test
- Time it takes to replicate measurable amount of virus
- Measure of viral load (between 10 and 40)
- lower number = more virus

Data Aggregation:

- $\approx 300,000$ observations (1.5 million with covariates)
- No or few tests on many days in many regions
- Time aggregation: weeks
- Space Aggregation: 9 regions

Brazil


6 9
7 1
1 3
8 2
2 4
3 5
4 6
9 8
5 7
region
Region1MT
Region2DF
Region3GO
Region4MG
Region5RJ
Region6North
Region7NorthEast
Region8CentralEast
Region9South



Region1MT
Region2DF
Region3GO
Region4MG
Region5RJ
Region6North
Region7NorthEast
Region8CentralEast
Region9South
Ct
Week
10
20
30
40

STATISTICAL MODEL

Indexes:

- $s = 1, ..., S$ for **s**pace
- $t = 1, ..., T$ for **t**ime
- $N_{s,t}$ **n**umber of measurements at (s, t)

Data model:

For all (s, t) $f_{s,t}$ is a random density.

$$X(s, t, i) \sim f_{s,t}, \qquad i = 1, ..., N_{s,t}.$$

Objective:

- Structural breaks in the $f_{s,t}$ (in time and space)

Building blocks

- **The centered log transformation**

$$\Psi : f \mapsto \log[f] - \frac{1}{|I|}\int_I \log[f(x)]dx$$

- Ψ is bijective and takes (positive) densities on I to $L_0^2(I)$
- This defines the Hilbert structure of the **Bayes space**

$$f \oplus g := \Psi^{-1}(\Psi[f] + \Psi[g]), \quad \|f\| := \left\{ \int_I \Psi[f](x)^2 dx \right\}^{1/2}.$$

Change point model:

- $$\Psi[f_{s,t}] = \varepsilon_{s,t} + \begin{cases} \mu_s^{(1)}, & \text{for} \quad t \leq \lfloor T\theta_s \rfloor \\ \mu_s^{(2)}, & \text{for} \quad t > \lfloor T\theta_s \rfloor. \end{cases}$$
- θ_s describes the time of the change
- A change occurs at s when $\mu_s^{(1)} \neq \mu_s^{(2)}$

Aim: Infer set of spatial locations

$$\mathcal{A}^{(changes)} := \{s : \ \mu_s^{(1)} \neq \mu_s^{(2)}\}$$

and times of changes.

Aim of inference

Recall

$$\mathcal{A}^{(changes)} := \{s : \ \mu_s^{(1)} \neq \mu_s^{(2)}\}.$$

Aim:

- Create an estimator $\widehat{\mathcal{A}}$ for $\mathcal{A}^{(changes)}$
- Consistency:

$$\mathbb{P}\big(\mathcal{A}^{(changes)} \subset \widehat{\mathcal{A}}\big) \to 1$$

- Level-α:

$$\mathbb{P}\big(\mathcal{A}^{(changes)} \neq \widehat{\mathcal{A}}\big) \to \alpha.$$

Our tools:

- An asymptotic test $\varphi[\mathcal{A}]$ of "no change in $\mathcal{A}$"

$$H_0 : \mathcal{A} \cap \mathcal{A}^{(changes)} = \emptyset$$

Change point model:

- Recall the data

$$X(s,t,i) \overset{i.i.d.}{\sim} f_{s,t}, \qquad i = 1, ..., N_{s,t}$$

- We consider kernel density estimators KDE

$$\hat{f}_{s,t}(x) := \frac{1}{N_{s,t} h_{s,t}} \sum_{i=1}^{N_{s,t}} K\Big(\frac{x - X(s,t,i)}{h_{s,t}}\Big)$$

- Consider $\Psi[\hat{f}_{s,t}]$ as proxies for $\Psi[f_{s,t}]$.

Sparsity (I)

Sparsity problem:

- $\hat{f}_{s,t} - f_{s,t}$ is not asymptotically negligible.
- $\hat{f}_{s,t}$ can be 0 and $\Psi(\hat{f}_{s,t})$ is not defined
- $\hat{f}_{s,t}$ may not even have the Fréchet mean

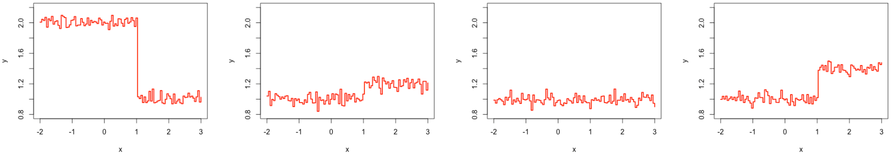
Remedies:

- Restriction to subinterval J, where $f_{s,t} \geq c > 0$.
- "Well-behaved version" $\breve{f}_{s,t}$ of $\hat{f}_{s,t}$ (equal with high probability)

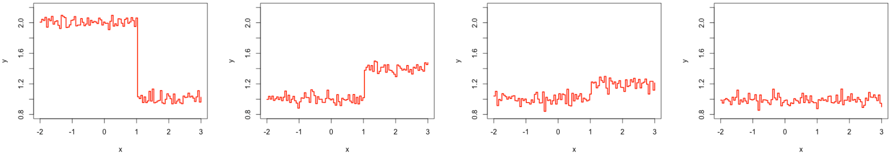
Mathematical details will follow.

SEQUENTIAL TESTING

Unordered locations:



Ordered locations s_1, s_2, s_3, s_4:



Sequential testing: Elimination

Statistically order locations $\hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{s}_4$ from the smallest to largest P-value of a change point test φ applied to each location.

This ordering is equal to the correct ordering s_1, s_2, s_3, s_4 with probability approaching 1.

Sequential testing: Set $\widehat{\mathcal{A}} = \emptyset$

1) $\varphi[s_1, ...s_4]$ (Is there a change in $\{s_1, ..., s_4\}$)?
   - Yes: Update $\widehat{\mathcal{A}} = \{s_1\}$ and move on
   - No: stop
2) $\varphi[s_2, ...s_4]$ (Is there still a change in $\{s_2, ..., s_4\}$)?
   - Yes: Update $\widehat{\mathcal{A}} = \{s_1, s_2\}$
   - No: stop
3) ...

Construction of φ:

- CUSUM statistic

$$\begin{aligned}\widehat{\Delta}[\mathcal{A}] &= \frac{1}{|\mathcal{A}|}\sum_{s\in\mathcal{A}}\frac{1}{T^2}\sum_{t=1}^{T}\Big\|\sum_{r=1}^{t}\Psi[\hat{f}_{s,r}] - \frac{t}{T}\sum_{r=1}^{T}\Psi[\hat{f}_{s,r}]\Big\|^2\\ &\approx \frac{1}{|\mathcal{A}|}\sum_{s\in\mathcal{A}}\frac{1}{T}\sum_{t=1}^{T}\Big\|\frac{1}{\sqrt{T}}\sum_{r=1}^{t}\Psi[\check{f}_{s,r}] - \frac{1}{\sqrt{T}}\frac{t}{T}\sum_{r=1}^{T}\Psi[\check{f}_{s,r}]\Big\|^2\\ &\xrightarrow{d} \frac{1}{|\mathcal{A}|}\sum_{s\in\mathcal{A}}\int_0^1 \|W_s(x) - xW_s(1)\|^2 dx.\end{aligned}$$

- $\{W_s\}_{s=1,\dots,S}$ is a Brownian motion in $\{L_0^2(J)\}^S$.
- $\{W_s\}_{s=1,\dots,S}$ has the same covariance as $\{\Psi(f_{s,1})\}_{s=1,\dots,S}$.

  Spatial dependence estimated, MC to get the distribution of the limit.

Theory (i)

Test decision:

$$\varphi_T[\mathcal{A}] := \begin{cases} 0, & \text{if} \quad \widehat{\Delta}[\mathcal{A}] \leq q_{1-\alpha}[\mathcal{A}], \\ 1, & \text{otherwise.} \end{cases}$$

On a set of probability approaching 1, $\check{\Delta}[\mathcal{A}] = \widehat{\Delta}[\mathcal{A}]$. These two random variables have the same asymptotic distribution.

On a set of probability approaching 1, for each subset of regions, we can detect change with probability approaching 1, if it occurs (consistency of $\varphi_T[\mathcal{A}]$).

We have a fixed (finite) number of regions.

Illustration of the main idea

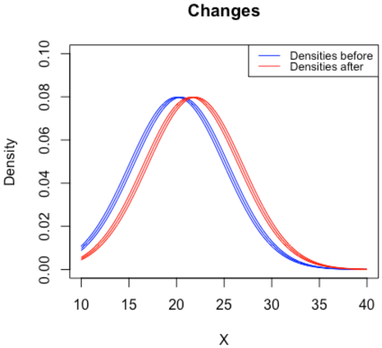
Replace asymptotic probability 1, by probability 1. Suppose $S = 4$, $\mathcal{A}^* = \{1, 2\}$ (change in 1 and 2, but not in 3 or 4).

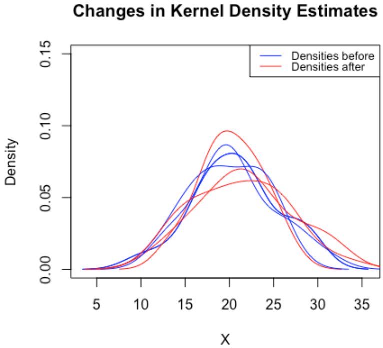
H_0^0: no change in $\{1, 2, 3, 4\}$. Change point detected with probability 1,
$\rightarrow$ test
H_0^1: no change in $\{2, 3, 4\}$. Change point detected with probability 1,
$\rightarrow$ test
H_0^2: no change in $\{3, 4\}$. Error with probability α (change in 3 or 4),
$\rightarrow$ test
H_0^3: no change in $\{4\}$. Error with probability α (change in 4),

$\widehat{\mathcal{A}} = \{1, 2, 3\} \iff$ reject H_0^2 $(\alpha) \rightarrow$ accept H_0^3 $(1 - \alpha)$
$\widehat{\mathcal{A}} = \{1, 2, 3, 4\} \iff$ reject H_0^2 $(\alpha) \rightarrow$ reject H_0^3 (α)

Probability of misidentifying $\mathcal{A}^* = \{1, 2\}$:
$\alpha(1 - \alpha) + \alpha\alpha = \alpha$.

Some simulations





Parameters: $T = 100, N = 50, \alpha = 0.05, S = 5$

	Small change	Large change
1 region	0.95 (0.82)	0.96 (0.82)
2 regions	0.90 (0.85)	0.95 (0.86)
3 regions	0.90 (0.89)	0.96 (0.91)

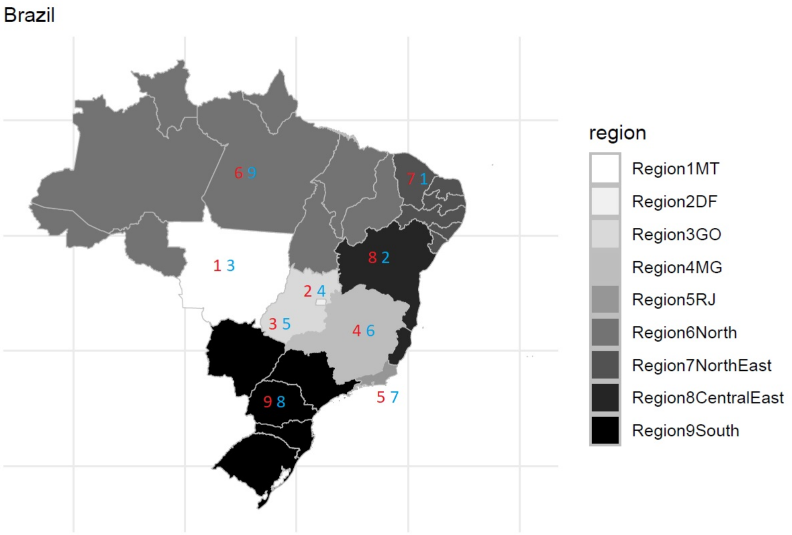
RESULTS (I)

Date	5-11 July of 2021	18-24 October of 2021	1-7 November of 2021
$t*$	62	77	79
year/week	2021/27	2021/42	2021/44
region	Region7NorthEast	Region8CentralEast	Region1MT
Date	1-7 November of 2021	1-7 November of 2021	15-21 November of 2021
$t*$	79	79	81
year/week	2021/44	2021/44	2021/46
region	Region2DF	Region3GO	Region4MG
Date	15-21 of November 2021	15-21 of November 2021	22-28 of November 2021
$t*$	81	81	82
year/week	2021/46	2021/46	2021/47
region	Region5RJ	Region9South	Region6North

Most changes in November 2021

Delta replaced by omicron in January 2022, with initial reports in November.

RESULTS


Brazil
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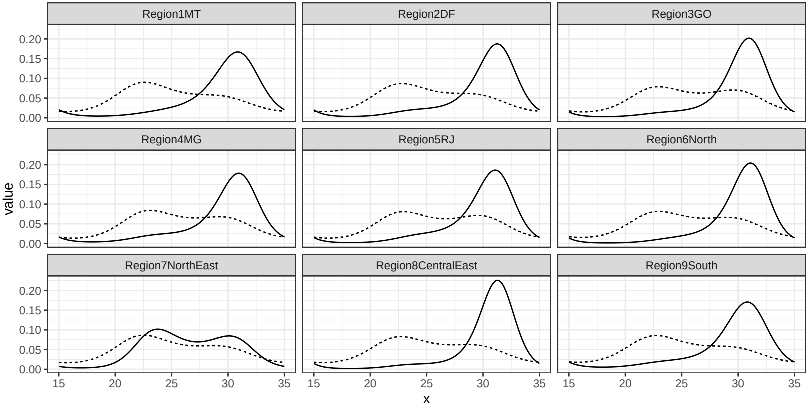
RESULTS (II)



Fréchet mean densities before (dashed) and after (continuous) the change points.

Omicron more infectious but less severe.

Theory (II)

Properties of $\breve{f}_{s,t}$: There exists an approximate version $((\breve{f}_{s,t})_s)_{t=1,\dots,T}$ of $((\hat{f}_{s,t})_s)_{t=1,\dots,T}$ such that

$$\mathbb{P}\big(\breve{f}_{s,t}(x) = \hat{f}_{s,t}(x),\ \forall x \in J,\ \forall s,t\big) \to 1, \quad as \quad T \to \infty, \qquad (1)$$

with $((\breve{f}_{s,t})_s)_{t=1,\dots,T}$ independent across t. The version satisfies for some fixed $c > 0$ the moment condition

$$\sup_{s,t} \mathbb{E}\|\Psi[\breve{f}_{s,t}]\|^4 \leq c \qquad (2)$$

and the mean approximation property

$$\|\mathbb{E}\Psi[\breve{f}_{s,t}] - \mathbb{E}\Psi[f_{s,t}]\| = o(T^{-1/2}). \qquad (3)$$

Theory (III)

$$\breve{G}_T(x) := \Big(\frac{1}{\sqrt{T}} \sum_{r=1}^{\lfloor xT \rfloor} \breve{Y}_{s,r} - \mathbb{E}[\breve{Y}_{s,r}]\Big)_s, \qquad x \in [0,1],$$

The $\breve{Y}_{s,r} = \Psi(\breve{f}_{s,t})$ are not identically distributed.

Proposition: Under our assumptions,

$$\{\breve{G}_T(x)\}_{x\in[0,1]} \xrightarrow{d} \{W(x)\}_{x\in[0,1]}$$

in the $J1$ topology of the Hilbert space $(L_0^2(J))^S$.